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New non-Lie ansätze and exact solutions of nonlinear reaction–diffusion–convection equations

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Abstract. A constructive method for obtaining new exact solutions of nonlinear evolution equations is further developed. The method is based on the consideration of a fixed nonlinear partial differential equation together with *an additional generating condition* in the form of a linear high-order ordinary differential equation. Using this method new non-Lie ansätze and exact solutions are obtained for two classes of diffusion equations with power and exponential nonlinearities, which describe real processes in physics, chemistry, and biology. The analysis of the found solutions and the relation of the proposed method to some approaches, which have been suggested in several recently published papers, are presented.

1. Introduction

Nonlinear second-order evolution equations (systems of equations) describe various processes in physics, chemistry and biology (heat and mass transfer, filtration of liquid, diffusion in chemical reactions etc). Construction of particular exact solutions for these equations remains an important problem. Finding exact solutions that have a physical, chemical or biological interpretation is of fundamental importance. The well known principle of linear superposition cannot be applied to generate new exact solutions to *nonlinear* partial differential equations (PDEs). Thus, the classical methods are not applicable for solving nonlinear PDEs. Of course, a change of variables can sometimes be found that transforms a nonlinear PDE into a linear equation, but finding exact solutions of most nonlinear PDEs generally requires new methods.

The most popular methods for construction of exact solutions to nonlinear PDEs are the method of inverse scattering and the Lie method. In this paper we do not consider the first one since it is only efficient for nonlinear PDEs with a very specific structure (see [1]). The Lie method [2–7] is based on using the Lie symmetry of a given PDE for the construction of its exact solutions. Although the technique of this method is well known, new results are constantly obtained for nonlinear PDEs with non-trivial Lie symmetries.

On the other hand it is well known that some very popular nonlinear PDEs have poor Lie symmetry. For example, the well known Fisher equation is invariant only under the time and space translations. The Lie method is not efficient for such PDEs since in these cases it enables one to construct ansätze and exact solutions, which can be obtained without using this cumbersome method. There are two ways to find a solution to this problem: (1), instead of PDEs with poor Lie symmetry, to find their analogues with the non-trivial

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Lie symmetry; (2), to suggest new approaches for construction of new ansätze and exact solutions that are not based on the search for Lie symmetry of a given PDE.

The first way was used in a series of our papers in collaboration with Fushchych [8–11] (see also [7]), in which all nonlinear generalizations of a classical heat (diffusion) equation that completely or partially preserve its Lie symmetry were described.

In the present paper we use the second way. Let us consider a nonlinear diffusion equation with a convection term of the form

$$T_t = [A(T)T_x]_x + B(T)T_x + C(T) \tag{1}$$

where $T = T(t, x)$ is the unknown function and $A(T), B(T), C(T)$ are arbitrary smooth functions. The indices t and x denote differentiation with respect to these variables.

Equation (1) generalizes a number of the well known nonlinear second-order evolution equations, describing various processes in physics [12], chemistry [13] and biology [14] (see [15] for a wide list of references).

It has been shown [16] that equation (1) by the local substitution

$$T \to U = \int A(T)\,\mathrm{d}T \equiv A_0(T) \tag{2}$$

is reduced to the form

$$U_{xx} = F_0(U)U_t + F_1(U)U_x + F_2(U) \tag{3}$$

where the functions $F_0(U)$, $F_1(U)$, $F_2(U)$ are easily determined via the formulae

$$F_0 = \frac{1}{A(T)}\Big|_{T=A_0^{-1}(U)} \qquad F_1 = -\frac{B(T)}{A(T)}\Big|_{T=A_0^{-1}(U)} \qquad F_2 = -C(T)\Big|_{T=A_0^{-1}(U)} \tag{4}$$

and A_0^{-1} is the inverse function to $A_0(T)$.

The Lie symmetry of equation (3) was completely described in [16]. According to the results of [16] any Lie ansatz, which reduces equation (3) to an ordinary differential equation (ODE), can be obtained by solving the following Lagrange system:

$$\frac{\mathrm{d}t}{\xi^0(t)} = \frac{\mathrm{d}x}{\xi^1(t,x)} = \frac{\mathrm{d}U}{\eta_0(t,x) + \eta_1(t,x)U} \tag{5}$$

where $\xi^0, \xi^1, \eta = \eta_0 + \eta_1 U$ are known coefficients of the infinitesimal operator

$$X = \xi^0\partial_t + \xi^1\partial_x + \eta\partial_U. \tag{6}$$

By solving equation (5) one can obtain a Lie ansatz of the form

$$U = g_0(t,x) + \varphi(\omega)g_1(t,x) \tag{7}$$

where $\varphi(\omega)$ is a new unknown function, $\omega = \omega(t, x)$ is the invariant variable, and $g_0(t, x)$ and $g_1(t, x)$ are fixed functions determined by ξ^0, ξ^1, η_0 and η_1. Note that the form (7) is the typical form for Lie ansätze of any quasilinear PDE (see, e.g. [7]). So substituting ansatz (7) into equation (3), one obtains a first- or second-order ODE, from which function $\varphi(\omega)$ can be found. Taking into account substitution (2), one can see that the general form of Lie ansätze for equation (1) is given by

$$A_0(T) = g_0(t,x) + \varphi(\omega)g_1(t,x). \tag{8}$$

Ansätze with structures differing from (7) and (8) are called *non-Lie ansätze* for PDEs of the forms (3) and (1), respectively. Of course, the form of non-Lie ansätze essentially depends on the form of a given PDE.

It turns out that it is possible to construct a set of non-Lie ansätze with principally different structure for some wide subclasses of nonlinear equations of the form (1) and (3).

With this in mind, an approach to the construction of non-Lie ansätze and exact solutions was suggested in [17, 18] that is based on the consideration of a given nonlinear PDE together with *an additional generating condition* in the form of a high-order ODE. This approach has been applied in [17–19] for obtaining solutions of some nonlinear evolution systems, which describe real processes in physics and chemistry.

In this paper, we study in detail two nonlinear equations of the form (1), namely

$$T_t = (T^\alpha T_x)_x + \lambda_1 T^\alpha T_x + \frac{1}{\alpha}(s_0 T^{1-\alpha} + s_1 T + q T^{1+\alpha}) \tag{9}$$

and

$$T_t = (\exp(\beta T) T_x)_x + \lambda_1 \exp(\beta T) T_x + \frac{1}{\beta}(s_0 \exp(-\beta T) + s_1 + q \exp(\beta T)) \tag{10}$$

where $T = T(t, x)$ is an unknown function and $\alpha\beta \neq 0, \lambda_1, s_0, s_1, q \in \mathbb{R}$. First, note that equation (9) is reduced by the substitution

$$U = T^\alpha \qquad \alpha \neq 0$$

to the equation

$$U_t = U U_{xx} + \frac{1}{\alpha} U_x^2 + \lambda_1 U U_x + s_0 + s_1 U + q U^2. \tag{11}$$

Analogously, the nonlinear evolution equation with exponential nonlinearities (10) is reduced by the substitution

$$U = \exp(\beta T) \qquad \beta \neq 0$$

to the equation

$$U_t = U U_{xx} + \lambda_1 U U_x + s_0 + s_1 U + q U^2. \tag{12}$$

Thus, hereinafter we consider the nonlinear equation

$$U_t = U U_{xx} + r U_x^2 + \lambda_1 U U_x + s_0 + s_1 U + q U^2 \tag{13}$$

which in the cases $r = \frac{1}{\alpha} \neq 0$ and $r = 0$ is locally equivalent to equations (9) and (10), respectively. In a particular case, any solution $U^*(t, x)$ of (13) generates a solution of the form

$$T^* = \begin{cases} (U^*)^{1/\alpha} & r \neq 0 \\ \dfrac{1}{\beta} \log U^* & r = 0 \end{cases} \tag{14}$$

to equations (9) and (10), respectively.

In section 2, the method of additional generating conditions is applied to the construction of new non-Lie ansätze of equations of the form (13). In section 3, new multiparameter families of solutions for nonlinear equations of the form (13) are constructed.

In section 4, new exact solutions of the nonlinear equations (9) and (10) are tabulated. Two types of the obtained solutions are applied for solving some nonlinear boundary-value problems.

Finally, in section 5 analysis of the found solutions, and the relationship of the proposed method with some approaches which have been suggested in several recently published papers, is presented.

2. A constructive method for obtaining non-Lie ansätze and new exact solutions of the nonlinear equation (13)

Here an approach to the construction of exact solutions is presented that is based on the consideration of a given nonlinear PDE together with an additional condition in the form of an ODE.

Consider the following class of nonlinear evolution second-order PDEs

$$U_t = \lambda_0 U U_{xx} + r U_x^2 + \lambda_1 U U_x + q U^2 + s_1 U + s_0 \tag{15}$$

where coefficients $\lambda_0, r, \lambda_1, q, s_1$ and s_0 are arbitrary constants. It is easily seen that the class of PDEs (15) contains the nonlinear equation (13) as a particular case.

If coefficients in (15) are arbitrary constants then this equation is invariant with respect to the translation transformations generated by operators

$$P_t = \frac{\partial}{\partial t} \qquad P_x = \frac{\partial}{\partial x} \tag{16}$$

and one can find plane wave solutions of the form

$$U = U(kx + vt) \qquad v, k \in \mathbb{R}. \tag{17}$$

But we do not construct such solutions as many papers have been devoted to the construction of plane wave solutions for various nonlinear PDEs of the form (1) (see, e.g. [15, 20, 21] and references therein).

Hereinafter we consider (15) together with the *additional generating conditions* in the form of linear high-order homogeneous equations, namely

$$\alpha_1(t,x)\frac{\mathrm{d}U}{\mathrm{d}x} + \cdots + \alpha_{m-1}(t,x)\frac{\mathrm{d}^{m-1}U}{\mathrm{d}x^{m-1}} + \frac{\mathrm{d}^m U}{\mathrm{d}x^m} = 0 \tag{18}$$

where $\alpha_1(t,x), \ldots, \alpha_{m-1}(t,x)$ are arbitrary smooth functions and the variable t is considered as a parameter. It is known that the general solution of (18) has the form

$$U = \varphi_0(t)g_0(t,x) + \cdots + \varphi_{m-1}(t)g_{m-1}(t,x) \tag{19}$$

where $\varphi_0(t), \varphi_1(t), \ldots, \varphi_{m-1}(t)$ are arbitrary functions and $g_0(t,x) = 1, g_1(t,x), \ldots, g_{m-1}(t,x)$ are fixed functions that form a fundamental system of solutions of (18). Note that in many cases the functions $g_1(t,x), \ldots, g_{m-1}(t,x)$ can be expressed in an explicit form in terms of elementary ones.

Let us consider relation (19) as an ansatz for PDEs of the form (15). It is important to note that this ansatz contains m yet-to-be determined functions $\varphi_i, i = 1, \ldots, m$. This enables us to reduce the given PDE of the form (15) to a quasilinear system of ODEs of the first order for the unknown functions φ_i. It is well known that such systems have been investigated in detail.

On the other hand equation (15) for $r = 0$ is a particular case of equation (3) and for $r \neq 0$ is locally equivalent to this equation. In fact, equation (15) for $r \neq 0$ is reduced by the local substitutions $t \to \lambda_0 t, \lambda_0 \neq 0$ and $U = T^\alpha, \alpha = 1/r$ (see the introduction) to equation (9) that is a particular case of equation (1) . Thus, the general forms of Lie ansätze for equation (15) are given by (7) or (8). Therefore ansatz (19) for $m \geqslant 2$ is just the non-Lie ansatz for any nonlinear equation of the form (15) since it has a structure different from (7) and (8).

Let us apply ansatz (19) to the equation (15). Indeed, calculating with the help of ansatz (19) the derivatives U_t, U_x, U_{xx} and substituting them into PDE (15), one obtains the following expression:

$$\begin{aligned}\varphi_{0,t}g_0 + \varphi_{1,t}g_1 + \cdots + \varphi_{m-1,t}g_{m-1} = {} & \varphi_0(s_1g_0 - g_{0,t}) + \cdots + \varphi_{m-1}(s_1g_{m-1} - g_{m-1,t}) \\ & +\varphi_0^2(\lambda_0 g_0 g_{0,xx} + r g_{0,x}^2 + \lambda_1 g_0 g_{0,x} + q g_0^2) \\ & + \cdots \varphi_{m-1}^2(\lambda_0 g_{m-1} g_{m-1,xx} + r g_{m-1,x}^2 + \lambda_1 g_{m-1} g_{m-1,x} + q g_{m-1}^2) \\ & +\varphi_0\varphi_1(\lambda_0 g_0 g_{1,xx} + \lambda_0 g_1 g_{0,xx} + \lambda_1 g_0 g_{1,x} + \lambda_1 g_1 g_{0,x} + 2r g_{0,x} g_{1,x} + 2q g_0 g_1) \\ & +\varphi_0\varphi_2(\lambda_0 g_0 g_{2,xx} + \lambda_0 g_2 g_{0,xx} + \lambda_1 g_0 g_{2,x} + \lambda_1 g_2 g_{0,x} + 2r g_{0,x} g_{2,x} + 2q g_0 g_2) \\ & + \cdots + \varphi_{m-2}\varphi_{m-1}(\lambda_0 g_{m-2} g_{m-1,xx} + \lambda_0 g_{m-1} g_{m-2,xx} + \lambda_1 g_{m-2} g_{m-1,x} \\ & +\lambda_1 g_{m-1} g_{m-2,x} + 2r g_{m-2,x} g_{m-1,x} + 2q g_{m-2} g_{m-1}) + s_0 \end{aligned} \tag{20}$$

where the indices t and x of functions $\varphi_i(t)$ and $g_i(t,x)$, $i = 0, 1, \ldots, m-1$, denote differentiation with respect to t and x. If one groups similar terms in accordance with the powers of the functions $\varphi_i(t)$, then sufficient conditions for reduction of this expression to a system of ODEs can be found. These sufficient conditions have the following form:

$$s_1 g_i - g_{i,t} = g_{i_1} Q_{ii_1}(t) \tag{21}$$

$$\lambda_0 g_i g_{i,xx} + r(g_{i,x})^2 + \lambda_1 g_i g_{i,x} + q(g_i)^2 = g_{i_1} R_{ii_1}(t) \tag{22}$$

$$\lambda_0(g_i g_{j,xx} + g_j g_{i,xx}) + 2r g_{i,x} g_{j,x} + \lambda_1(g_i g_{j,x} + g_j g_{i,x}) + 2q g_i g_j = g_{i_1} T_{ij}^{i_1}(t) \qquad i < j \tag{23}$$

where coefficients $Q_{ii_1}, R_{ii_1}, T_{ij}^{i_1}$ on the right-hand side are defined by the expressions on the left-hand side.

With the help of conditions (21)–(23), the following system of ODEs is obtained

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = Q_{i_1 i}\varphi_{i_1} + R_{i_1 i}(\varphi_{i_1})^2 + T_{i_1 i_2}^{i}\varphi_{i_1}\varphi_{i_2} + \delta_{i,0}s_0 \tag{24}$$

to find the unknown functions $\varphi_i, i = 0, \ldots, m-1$. In the right-hand sides of relations (21)–(23) and (24), a summation is assumed from 0 to $m-1$ over the repeated indices i_1, i_2 and $\delta_{i,0} = 0, 1$ is the Kronecker symbol. So, we have obtained the following statement.

Theorem 1. Any solution of system (24) generates the exact solution in the form (19) for nonlinear PDE (15), if the functions $g_i, i = 0, \ldots, m-1$ satisfy conditions (21)–(23).

Remark 1. Equation (15) for $\lambda_0 = 0$ can be considered as a generalization of the Hamilton–Jacobi equation. It is well known that the problem of construction of exact solutions in the *explicit form* for the Hamilton–Jacobi equation is a non-trivial one (see, e.g. [7]). The suggested method can be applied to the construction of explicit solutions of equation (15) for $\lambda_0 = 0$ too. In the case $\lambda_0 \neq 0$, equation (15) is reduced to the form (13) by the local substitution $t \to \lambda_0 t$.

Remark 2. The suggested method can be realized for systems of PDEs (see the examples in [17–19]) and for PDEs with derivatives of second or higher orders with respect to t and x. In the last case one will obtain systems of ODEs of second or higher orders.

Remark 3. If the coefficients $\lambda_0, \lambda_1, s_0, s_1$ and q in equation (15) are smooth functions of the variable t then one can also construct families of exact solutions. But in this case the systems of ODEs with time-dependent coefficients are obtained.

Since we suggest a constructive method for finding new non-Lie ansätze and exact solutions, its efficiency will be shown by the examples below.

In fact, let us use theorem 1 for the construction of non-Lie solutions of equation (13). Consider an additional generating condition of the third order of the form

$$\alpha_1(t)\frac{\mathrm{d}U}{\mathrm{d}x} + \alpha_2(t)\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + \frac{\mathrm{d}^3U}{\mathrm{d}x^3} = 0 \tag{25}$$

which is the particular case of (18) for $m = 3$. Condition (25) generates the following chain of the ansätze:

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma_1(t)x) + \varphi_2(t)\exp(\gamma_2(t)x) \tag{26}$$

if $\gamma_{1,2}(t) = \frac{1}{2}(\pm(\alpha_2^2 - 4\alpha_1)^{1/2} - \alpha_2)$ and $\gamma_1 \neq \gamma_2$;

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma(t)x) + x\varphi_2(t)\exp(\gamma(t)x) \tag{27}$$

if $\gamma_1 = \gamma_2 = \gamma \neq 0$;

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)\exp(\gamma(t)x) \tag{28}$$

if $\alpha_1 = 0$;

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 \tag{29}$$

if $\alpha_1 = \alpha_2 = 0$.

Remark 4. In the case $D = \alpha_2^2 - 4\alpha_1 < 0$, one obtains complex functions $\gamma_1 = \gamma_2^* = \frac{1}{2}(\pm\mathrm{i}(-D)^{1/2} - \alpha_2)$, $\mathrm{i}^2 = -1$ and then ansatz (26) is reduced to the form

$$U = \varphi_0(t) + [\psi_1(t)\cos(\tfrac{1}{2}(-D)^{1/2}x) + \psi_2(t)\sin(\tfrac{1}{2}(-D)^{1/2}x)]\exp\left(-\frac{\alpha_2 x}{2}\right) \tag{30}$$

where $\varphi_0(t), \psi_1(t), \psi_2(t)$ are yet-to-be determined functions.

By substituting the functions $g_0 = 1$, $g_1 = \exp(\gamma_1(t)x)$, $g_2 = \exp(\gamma_2(t)x)$ from ansatz (26) into relations (21)–(23), one can obtain

$$\begin{gathered} Q_{00} = Q_{11} = Q_{22} = s_1 \qquad R_{00} = q \\ T_{12}^0 = \frac{4qr}{1+r} \qquad T_{01}^1 = T_{02}^2 = q\left(1 + \frac{r}{1+r}\right) \end{gathered} \tag{31}$$

and the following relations

$$R_{ii_1} = Q_{ii_1} = T_{ij}^{i_1} = 0 \tag{32}$$

for all combinations of the indices $i, i_1, j = 0, 1, 2$ and $k = 1, 2$ not listed in (31). Simultaneously the following constraints spring up:

$$r = 0 \qquad \gamma_{1,2}(t) = \tfrac{1}{2}(\pm(\lambda_1^2 - 4q)^{1/2} - \lambda_1) \tag{33}$$

or

$$\lambda_1 = 0 \qquad \gamma_{1,2}(t) = \pm(-q/(1+r))^{1/2} \qquad r \neq -1. \tag{34}$$

With the help of coefficients (31), (32), system (24) is reduced to the form

$$\begin{aligned} \frac{\mathrm{d}\varphi_0}{\mathrm{d}t} &= q\varphi_0^2 + s_1\varphi_0 + s_0 \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}t} &= s_1\varphi_1 + q\varphi_0\varphi_1 \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}t} &= s_1\varphi_2 + q\varphi_0\varphi_2 \end{aligned} \tag{35}$$

in the case of constraint (33) and to the form

$$\begin{aligned}
\frac{d\varphi_0}{dt} &= q\varphi_0^2 + s_1\varphi_0 + s_0 + \frac{4rq}{1+r}\varphi_1\varphi_2 \\
\frac{d\varphi_1}{dt} &= s_1\varphi_1 + q\left(1 + \frac{r}{1+r}\right)\varphi_0\varphi_1 \\
\frac{d\varphi_2}{dt} &= s_1\varphi_2 + q\left(1 + \frac{r}{1+r}\right)\varphi_0\varphi_2
\end{aligned} \tag{36}$$

in the case of constraint (34).

By substituting the functions $g_0 = 1$, $g_1 = \exp(\gamma(t)x)$, $g_2 = x\exp(\gamma(t)x)$ from ansatz (27) into relations (21)–(23), the corresponding values of the functions R_{ii_1}, Q_{ii_1}, $T^j_{ii_1}$, are obtained, for which system (24) coincides with the system of ODEs (35). But in this case we obtain the constraints $r = 0$, $\lambda_1^2 - 4q = 0$ and $\gamma = -\lambda_1/2$.

Similarly, we obtain with the help of ansatz (28) the following system of ODEs

$$\begin{aligned}
\frac{d\varphi_0}{dt} &= \lambda_1\varphi_0\varphi_1 + s_1\varphi_0 + s_0 \\
\frac{d\varphi_1}{dt} &= \lambda_1\varphi_1^2 + s_1\varphi_1 \\
\frac{d\varphi_2}{dt} &= \lambda_1\varphi_1\varphi_2 + s_1\varphi_2
\end{aligned} \tag{37}$$

to find the unknown functions $\varphi_i, i = 0, 1, 2$. In this case we obtain the constraints $r = q = 0$ and $\gamma = -\lambda_1$.

Finally, ansatz (29) for $\lambda_1 = q = 0$ gives the following system of ODEs

$$\begin{aligned}
\frac{d\varphi_0}{dt} &= 2\varphi_0\varphi_2 + s_1\varphi_0 + s_0 + r\varphi_1^2 \\
\frac{d\varphi_1}{dt} &= (2 + 4r)\varphi_1\varphi_2 + s_1\varphi_1 \\
\frac{d\varphi_2}{dt} &= (2 + 4r)\varphi_2^2 + s_1\varphi_2.
\end{aligned} \tag{38}$$

Thus, any ansatz from set (26)–(29) can be applied for the reduction of the nonlinear PDE (13) to a system of first-order ODEs. But in all cases additional constraints for coefficients $r, \lambda_1, s_0, s_1, q \in \mathbb{R}$ spring up. These constraints follow from relations (21)–(23) and they are necessary for the above-mentioned reduction.

On the other hand, theorem 1 gives only *sufficient* conditions for such reduction. In some cases, noting additional relations between the fundamental system elements $g_0(t, x) = 1, g_1(t, x), \ldots, g_{m-1}(t, x)$, it is possible to find simpler sufficient conditions for the above-mentioned reduction. Let us give an example in the case of ansatz (28). Noting that $g_{2,t} = \gamma^{-2}\frac{d\gamma}{dt}g_1 g_{2,xx}$, one can transfer this term from relations (21), $i = 2$ into (23), $i = 1$, $j = 2$, so that two relations in (21)–(23) take new form. Taking into account this circumstance for ansatz (28), one can find at once the case $r = -1$ and $\lambda_1 = q = 0$

when the nonlinear PDE (13) is reduced to the system of first-order ODEs

$$\begin{aligned}\frac{d\gamma}{dt} &= \gamma^2\varphi_1\\ \frac{d\varphi_0}{dt} &= -\varphi_1^2 + s_1\varphi_0 + s_0\\ \frac{d\varphi_1}{dt} &= s_1\varphi_1\\ \frac{d\varphi_2}{dt} &= -2\gamma\varphi_1\varphi_2 + s_1\varphi_2 + \gamma^2\varphi_0\varphi_2.\end{aligned} \tag{39}$$

It is easily seen that in this case the function $\gamma(t) \neq$ constant and the first equation in (39) is an additional condition for obtaining the function γ.

By analogy with the additional generating condition (25), we can consider the fourth-order condition of the form

$$\alpha_1(t)\frac{dU}{dx} + \alpha_2(t)\frac{d^2U}{dx^2} + \alpha_3(t)\frac{d^3U}{dx^3} + \frac{d^4U}{dx^4} = 0. \tag{40}$$

This condition generates a chain of the ansätze. Although there are other interesting cases, probably the most non-trivial one occurs when we will consider the ansatz

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma_1 x) + \varphi_2(t)\exp(\gamma_2 x) + \varphi_3(t)\exp(\gamma_3 x) \tag{41}$$

where $\varphi_0(t), \ldots, \varphi_3(t)$ are yet-to-be determined functions and $\gamma_1, \gamma_2, \gamma_3$ are some *unequal* constants and $\gamma_1\gamma_2\gamma_3 \neq 0$. It turns out that it is possible to reduce equation (13) with the help of (41) to a system of ODEs of the form (24) only in a special case. Indeed, by substituting the functions $g_0 = 1$, $g_1 = \exp(\gamma_1 x)$, $g_2 = \exp(\gamma_2 x)$ and $g_3 = \exp(\gamma_3 x)$ from ansatz (41) into relations (21) at $i = 0, 1, 2, 3$, one obtains

$$Q_{00} = Q_{11} = Q_{22} = Q_{33} = s_1 \qquad Q_{ii_1} = 0, i \neq i_1. \tag{42}$$

Substitution of the functions g_i, $i = 0, \ldots, 3$, into relation (22) at $i = 0$ gives

$$R_{00} = q \qquad R_{0a} = 0 \qquad a = 1, 2, 3. \tag{43}$$

In the cases $i = 1, 2, 3$ relations (22) take the following form:

$$[(1+r)\gamma_a^2 + \lambda_1\gamma_a + q]\exp(2\gamma_a x) = g_{i_1}R_{ai_1}(t) \qquad a = 1, 2, 3 \tag{44}$$

where a summation is assumed from 0 to 3 over the repeated indices i_1. It turns out that these relations can be fulfilled only in the case (i)

$$\gamma_{1,2} = \frac{\pm(\lambda_1^2 - 4q(1+r))^{1/2} - \lambda_1}{2(1+r)} \qquad \gamma_3 = \tfrac{1}{2}\gamma_2 \qquad (\lambda_1^2 - 4q(1+r))(1+r) \neq 0 \tag{45}$$

or (ii)

$$\gamma_2 = \tfrac{1}{2}\gamma_1 \qquad \gamma_3 = \tfrac{1}{4}\gamma_1 \tag{46}$$

where γ_1 is a root of the quadratic equation $(1+r)\gamma_a^2 + \lambda_1\gamma_a + q = 0$. It is easy to check that constraint (46) is too strong because it is impossible to satisfy relations (23). Therefore, the first case only is considered below.

Using relations (44) under constraint (45), one obtains

$$R_{32} = \frac{\lambda_1\gamma_2 + 3q}{4} \qquad R_{ai_1} = 0 \qquad a = 1, 2, 3 \tag{47}$$

where the combinations of the indices $(a, i_1) \neq (3, 2)$.

Similarly, relations (23) at $(i, j) = (1, 2)$ and $(i, j) = (2, 3)$ can be fulfilled only in the case $\gamma_1 + \gamma_2 = \gamma_3$ (i.e. $q = -6\lambda_1^2 \neq 0$) and $r = -\frac{2}{3}$ (i.e. $\gamma_1 = 3\lambda_1$). Under such constraints, it is easy to construct the coefficients $T_{ij}^{i_1}$, namely:

$$T_{02}^2 = 18\lambda_1^2 \qquad T_{03}^3 = -6\lambda_1^2 \qquad T_{13}^0 = 18\lambda_1^2 \qquad T_{12}^3 = 54\lambda_1^2 \tag{48}$$

and

$$T_{ij}^{i_1} = 0 \tag{49}$$

for all combinations of the indices $i, i_1, j = 0, 1, 2, 3$ not listed in (48).

With the help of coefficients (42), (43), (47)–(49), system (24) is reduced to the form

$$\begin{aligned}
\frac{d\varphi_0}{dt} &= s_1\varphi_0 + 18\lambda_1^2\varphi_1\varphi_3 - 6\lambda_1^2\varphi_0^2 + s_0 \\
\frac{d\varphi_1}{dt} &= s_1\varphi_1 \\
\frac{d\varphi_2}{dt} &= s_1\varphi_2 + 18\lambda_1^2\varphi_0\varphi_2 - 6\lambda_1^2\varphi_3^2 \\
\frac{d\varphi_3}{dt} &= s_1\varphi_3 - 6\lambda_1^2\varphi_0\varphi_3 + 54\lambda_1^2\varphi_1\varphi_2.
\end{aligned} \tag{50}$$

Thus, according to theorem 1 any solution of the ODE system (50) generates an exact solution of the form

$$U = \varphi_0(t) + \varphi_1(t)\exp(3\lambda_1 x) + \varphi_2(t)\exp(-6\lambda_1 x) + \varphi_3(t)\exp(-3\lambda_1 x) \tag{51}$$

for nonlinear equation

$$U_t = UU_{xx} - \tfrac{2}{3}U_x^2 + \lambda_1 UU_x + s_0 + s_1 U - 6\lambda_1^2 U^2. \tag{52}$$

It is easily seen that this equation is locally equivalent to the nonlinear diffusion equation with the convection term

$$T_t = (T^{-\frac{3}{2}}T_x)_x + \lambda_1 T^{-\frac{3}{2}}T_x - \tfrac{2}{3}(s_0 T^{\frac{5}{2}} + s_1 T - 6\lambda_1^2 T^{-\frac{1}{2}}). \tag{53}$$

Therefore any solution of (52) can be transformed into a solution of the nonlinear heat equation (53), using (14) for $\alpha = -\frac{3}{2}$.

3. Construction of the families of non-Lie exact solutions of the nonlinear equation (13)

The systems of ODEs, which were obtained in the previous section, enable us to construct the multiparametric families of non-Lie exact solutions of the nonlinear equation (13). Having obtained the solutions of equation (13), we can easily construct solutions for the nonlinear equations (9) and (10) (see substitution (14)).

So consider the system of ODEs (35). The first equation in (35) is autonomous and its solutions essentially depend on the coeficients q, s_1, s_0. By solving this equation we obtain the following solutions [22]

$$\varphi_0 = \frac{1}{2q}\begin{cases} \dfrac{2}{c_0 - t} - s_1 & \text{if } D = 0 \\ \sqrt{(-D)}\tan\left(\dfrac{\sqrt{(-D)}}{2}(t - c_0)\right) - s_1 & \text{if } D < 0 \\ \sqrt{D}\coth\left(\dfrac{\sqrt{D}}{2}(c_0 - t)\right) - s_1 & \text{if } (2q\varphi_0 + s_1)^2 > D > 0 \\ \sqrt{D}\tanh\left(\dfrac{\sqrt{D}}{2}(c_0 - t)\right) - s_1 & \text{if } D > (2q\varphi_0 + s_1)^2 > 0 \end{cases} \tag{54}$$

where $D = s_1^2 - 4s_0q$.

Having solution (54), it is easy to find the general solution for the system of ODEs (35). So, ansatz (26) generates the following three-parameter family of solutions of equation (13) at $r = 0$

$$U = \varphi_0(t) + \frac{1}{\mu(t)}\left[c_1\exp\left(\frac{1}{2}s_1t + \gamma_1x\right) + c_2\exp\left(\frac{1}{2}s_1t + \gamma_2x\right)\right] \tag{55}$$

where

$$\mu(t) = \begin{cases} |c_0 - t| & \text{if } D = 0 \\ \left|\cos\left[\dfrac{\sqrt{(-D)}}{2}(t - c_0)\right]\right| & \text{if } D < 0 \\ \left|\sinh\left[\dfrac{\sqrt{D}}{2}(c_0 - t)\right]\right| & \text{if } (2q\varphi_0 + s_1)^2 > D > 0 \\ \cosh\left[\dfrac{\sqrt{D}}{2}(c_0 - t)\right] & \text{if } D > (2q\varphi_0 + s_1)^2 > 0 \end{cases} \tag{56}$$

and $\gamma_{1,2} = \frac{1}{2}(\pm(\lambda_1^2 - 4q)^{1/2} - \lambda_1)$. In (54)–(56) and hereinafter c_0, c_1, c_2 are arbitrary constants. Note that in the case $D > 0$ the family of solutions (55) can be written in the form

$$U = \varphi_0(t) + \frac{1}{a_0 \pm \exp\sqrt{D}t}\left[c_1\exp\left(\frac{s_1 + \sqrt{D}}{2}t + \gamma_1x\right) + c_2\exp\left(\frac{s_1 + \sqrt{D}}{2}t + \gamma_2x\right)\right] \tag{57}$$

where $a_0 = \exp(c_0\sqrt{D})$.

By solving the system of ODEs (36), we obtain the following family of solutions of equation (13) for $\lambda_1 = 0, r \neq -1$

$$U = \varphi_0(t) + \exp\left[s_1t + q\frac{2r+1}{r+1}\int\varphi_0(t)\,\mathrm{d}t\right]\left[c_1\exp\left(-\sqrt{\frac{-q}{1+r}}x\right) + c_2\exp\left(\sqrt{\frac{-q}{1+r}}x\right)\right] \tag{58}$$

where $\varphi_0(t)$ is an arbitrary solution of the integro-differential equation

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}t} = q\varphi_0^2 + s_1\varphi_0 + s_0 + 4c_1c_2\frac{qr}{1+r}\exp\left[2s_1t + 2q\left(1 + \frac{r}{r+1}\right)\int\varphi_0(t)\,\mathrm{d}t\right]. \tag{59}$$

This equation is reduced to a nonlinear second-order ODE that cannot be integrated in the general case. However, it is easily seen that in the case $c_1c_2 = 0$, integro-differential equation (59) is reduced to the first ODE in system (35). So, two-parameter families of solutions

$$U = \varphi_0(t) + \frac{c_2}{(\mu(t))^{\frac{2r+1}{r+1}}} \exp\left(\frac{1}{2}s_1 t + \sqrt{\frac{-q}{1+r}}x\right) \tag{60}$$

and

$$U = \varphi_0(t) + \frac{c_1}{(\mu(t))^{\frac{2r+1}{r+1}}} \exp\left(\frac{1}{2}s_1 t - \sqrt{\frac{-q}{1+r}}x\right) \tag{61}$$

are obtained for $c_1 = 0$ and $c_2 = 0$, respectively (the functions $\varphi_0(t)$ and $\mu(t)$ are defined in (54) and (56)).

Remark 5. In the case $r = -\frac{1}{2}$, integro-differential equation (59) is also reduced to an ODE. However, the general solution of this ODE cannot be obtained in the explicit form since the obtained ODE is the Riccati-type equation.

It is very important to note that in the case $\lambda_1 = 0$, $q/(1+r) > 0$, $r \neq -1$, we can construct periodic solutions of the equation (13). In fact, it is easily seen that, if complex constants $2c_1 = c_{10} - \mathrm{i}c_{11}, 2c_2 = c_{10} + \mathrm{i}c_{11}$, than we obtain from (58) the following three-parameter family of solutions of equation (13)

$$U = \varphi_0(t) + \exp\left[s_1 t + q\frac{2r+1}{r+1}\int \varphi_0(t)\,\mathrm{d}t\right]\left[c_{10}\cos\left(\sqrt{\frac{q}{1+r}}x\right) + c_{11}\sin\left(\sqrt{\frac{q}{1+r}}x\right)\right] \tag{62}$$

where c_{10}, c_{11} are arbitrary real constants and $\varphi_0(t)$ is an arbitrary solution of the integro-differential equation

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}t} = q\varphi_0^2 + s_1\varphi_0 + s_0 + (c_{10}^2 + c_{11}^2)\frac{qr}{1+r}\exp\left[2s_1 t + 2q\left(1 + \frac{r}{r+1}\right)\int \varphi_0(t)\,\mathrm{d}t\right]. \tag{63}$$

The ansatz (27) generates the three-parameter family of solutions of equation (13) at $\lambda_1^2 - 4q = 0$ with the similar structure, namely:

$$U = \varphi_0(t) + \frac{1}{\mu(t)}\left[c_1\exp\left(\frac{1}{2}(s_1 t - \lambda_1 x)\right) + c_2 x\exp\left(\frac{1}{2}(s_1 t - \lambda_1 x)\right)\right] \tag{64}$$

where the functions φ_0 and μ are presented in formulae (54) and (56).

Similarly we obtain the general solution of system (37)

$$\begin{aligned}
\varphi_0(t) &= \frac{\exp(\frac{1}{2}s_1 t)}{\mu_1(t)}\left(c_0 + s_0\int \frac{\mu_1(t)}{\exp(\frac{1}{2}s_1 t)}\,\mathrm{d}t\right)\\
\varphi_1 &= \frac{1}{2\lambda_1}\begin{cases} \dfrac{2}{c_1 - t} & \text{if } s_1 = 0\\ |s_1|\coth\left(\dfrac{|s_1|}{2}(c_1 - t)\right) - s_1 & \text{if } (2\lambda_1\varphi_1 + s_1) > |s_1| > 0\\ |s_1|\tanh\left(\dfrac{|s_1|}{2}(c_1 - t)\right) - s_1 & \text{if } |s_1| > (2\lambda_1\varphi_1 + s_1) > 0\end{cases}\\
\varphi_2(t) &= c_2\frac{\exp\left(\frac{1}{2}s_1 t\right)}{\mu_1(t)}
\end{aligned} \tag{65}$$

where

$$\mu_1(t) = \begin{cases} |c_1 - t| & \text{if } s_1 = 0 \\ \left|\sinh\left[\frac{|s_1|}{2}(c_1 - t)\right]\right| & \text{if } (2\lambda_1\varphi_1 + s_1) > |s_1| > 0 \\ \cosh\left[\frac{|s_1|}{2}(c_1 - t)\right] & \text{if } |s_1| > (2\lambda_1\varphi_1 + s_1) > 0. \end{cases} \tag{66}$$

So ansatz (28) generates the following three-parameter family of solutions of equation (13) at $r = q = 0, \lambda_1 \neq 0$

$$U = \frac{\exp(\frac{1}{2}s_1 t)}{\mu_1(t)}\left(c_0 + s_0 \int \frac{\mu_1(t)}{\exp(\frac{1}{2}s_1 t)}\,\mathrm{d}t\right) + \varphi_1(t)x + \frac{c_2}{\mu_1(t)} \exp\left(\frac{1}{2}s_1 t - \lambda_1 x\right) \tag{67}$$

where the functions φ_1 and μ_1 are presented in formulae (65) and (66).

System (39) contains the subsystem

$$\begin{aligned} \frac{\mathrm{d}\gamma}{\mathrm{d}t} &= \gamma^2\varphi_1 \\ \frac{\mathrm{d}\varphi_0}{\mathrm{d}t} &= -\varphi_1^2 + s_1\varphi_0 + s_0 \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}t} &= s_1\varphi_1 \end{aligned} \tag{68}$$

that is integrated and has a general solution in the explicit form

$$\begin{aligned} \gamma &= (\gamma_0 - c_1 s_1^{-1} \exp s_1 t)^{-1} \\ \varphi_0 &= \frac{1}{s_1}(-s_0 + c_0 s_1 \exp s_1 t - c_1^2 \exp 2s_1 t) \\ \varphi_1 &= c_1 \exp s_1 t \end{aligned} \tag{69}$$

if $s_1 \neq 0$ and

$$\begin{aligned} \gamma &= (\gamma_0 - c_1 t)^{-1} \\ \varphi_0 &= (s_0 - c_1^2)t + c_0 \\ \varphi_1 &= c_1 \end{aligned} \tag{70}$$

if $s_1 = 0$ and γ_0 is an arbitrary constant. Then the following ODEs

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = \left[s_1 - 2c_1\gamma \exp s_1 t + \frac{\gamma^2}{s_1}(-s_0 + c_0 s_1 \exp s_1 t - c_1^2 \exp 2s_1 t)\right]\varphi_2 \tag{71}$$

and

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = [\gamma^2(c_0 + (s_0 - c_1^2)t) - 2\gamma c_1]\varphi_2 \tag{72}$$

for finding $\varphi_2(t)$ are obtained, respectively.

So, ansatz (28) generates the following family of solutions of equation (13) for $\lambda_1 = q = 0, r = -1$:

$$U = \frac{1}{s_1}(-s_0 + c_0 s_1 \exp s_1 t - c_1^2 \exp 2s_1 t) + c_1 x \exp s_1 t + \varphi_2(t) \exp\left(\frac{x}{\gamma_0 - c_1 s_1^{-1} \exp s_1 t}\right) \tag{73}$$

if $s_1 \neq 0$ and

$$U = c_0 + (s_0 - c_1^2)t + c_1 x + \varphi_2(t)\exp\left(\frac{x}{\gamma_0 - c_1 t}\right) \tag{74}$$

if $s_1 = 0$. In (73) and (74), the function φ_2 is a solution of linear ODEs (71) and (72), respectively.

Finally, by solving system (38), one can make sure that ansatz (29) generates the following three-parameter family of solutions of equation (13) at $\lambda_1 = q = 0$

$$\begin{aligned} U = \exp\left(s_1 t + 2\int \varphi_2(t)\,\mathrm{d}t\right)\Bigg[c_0 + \int\Bigg[s_0 \exp\left(-s_1 t - 2\int \varphi_2(t)\,\mathrm{d}t\right) \\ + r c_1^2 \exp\left(s_1 t + 2(1+4r)\int \varphi_2(t)\,\mathrm{d}t\right)\Bigg]\mathrm{d}t\Bigg] \\ + c_1 x \exp\left(s_1 t + (2+4r)\int \varphi_2(t)\,\mathrm{d}t\right) + \varphi_2(t)x^2 \end{aligned} \tag{75}$$

where $s_1 \neq 0$ and

$$\varphi_2 = \begin{cases} \dfrac{c_2 s_1 \exp s_1 t}{(2+4r)(1 - c_2 \exp s_1 t)} & \text{if } r \neq -\frac{1}{2} \\ c_2 \exp s_1 t & \text{if } r = -\frac{1}{2}. \end{cases} \tag{76}$$

In the case $s_1 = 0$, we obtain the family of the solutions

$$\begin{aligned} U = (c_2 - t)^{-1}\left[c_0|c_2 - t|^{\frac{2r}{1+2r}} - s_0\frac{1+2r}{2+2r}(c_2 - t)^2 + \frac{1+2r}{2}\left(c_1 + \frac{x}{1+2r}\right)^2\right] \\ r \neq -1, -\tfrac{1}{2} \end{aligned} \tag{77}$$

to the equation

$$U_t = U U_{xx} + r U_x^2 + s_0. \tag{78}$$

In the cases $r = -1$ and $r = -\frac{1}{2}$, the solutions

$$U = s_0(t - c_2)\log|t - c_2| + c_0(t - c_2) + \frac{(c_1 + x)^2}{2(t - c_2)} \tag{79}$$

and

$$U = c_0 \exp(2c_2 t) + c_2\left(x + \frac{c_1}{2c_2}\right)^2 - \frac{s_0}{2c_2} \tag{80}$$

are obtained, respectively.

The family of solutions (77) for $c_1 = 0$ gives the exact solution

$$U = c_0(c_2 - t)^{\frac{-1}{1+2r}} - s_0\frac{1+2r}{2+2r}(c_2 - t) + \frac{x^2}{(2+4r)(c_2 - t)}. \tag{81}$$

On the other hand, (78) and (81) are reduced by the substitution

$$T(\tau, x) = U^{\frac{1}{\mu-1}} \qquad \tau = \frac{t}{\mu} \qquad r = \frac{1}{\mu - 1} \qquad s_0 = \frac{b(1-\mu)}{\mu} \tag{82}$$

to the equation

$$T_\tau = (T^\mu)_{xx} - bT^{2-\mu} \qquad \mu \neq 0, 1 \tag{83}$$

and to the solution

$$T^{\mu-1} = c_0(c_2 - \mu\tau)^{\frac{1-\mu}{1+\mu}} + \frac{b(\mu^2-1)}{2\mu^2}(c_2 - \mu\tau) + \frac{\mu-1}{2(1+\mu)}\frac{x^2}{(c_2-\mu\tau)} \qquad (84)$$

respectively. This solution was obtained in [24] as a generalization of the known Barenblatt–Zeldovich solution (see (84) for $b = 0$) [23] for the nonlinear heat equation

$$T_\tau = (T^\mu)_{xx} \qquad \mu \neq 0, 1. \qquad (85)$$

It is easily seen that solutions (55), (58), (60)–(62), (64), (67), (73), and (74) are not of the form (17). Therefore, if the maximal algebra of invariance of equation (13) is the two-dimensional algebra (16) then they are just non-Lie solutions of this nonlinear equation. Taking into account the results of papers [16, 28], it is easily seen that the nonlinear equation (13) is invariant with respect to the trivial algebra (16) if (i) $\lambda_1 s_0 s_1 \neq 0$, (ii) $\lambda_1 = 0, s_0 s_1 \neq 0$ or (iii) $\lambda_1 = 0, s_0 q \neq 0$ (other coefficients are arbitrary parameters). In these cases the above-found solutions cannot be obtained using the Lie method.

Of course, if some coefficients vanish in equation (13) then one can obtain the nonlinear equation with the three-, four- or five-dimensional Lie algebra (for details see [16, 28]). For such equations, one has to prove additionally that the constructed solutions are non-Lie solutions. For example, equation (83) is invariant with respect to the 3-dimensional Lie algebra [28]. Generally speaking, it has been additionally proved that solution (84) cannot be obtained using the classical Lie procedure (see, e.g. [2–7]). Note that the Barenblatt–Zeldovich solution is a similarity solution and it can be obtained using the Lie symmetry of equation (85).

It turns out that it is a non-trivial problem to construct just non-Lie exact solutions for nonlinear PDEs with non-trivial Lie symmetries. Indeed, a new non-Lie ansatz does not guarantee construction of new non-Lie exact solutions, if a nonlinear PDE has a non-trivial symmetry. For example, let us consider equation (13) when $r = 0, s_0 = 0, q = \frac{2}{9}\lambda_1^2 \neq 0$, i.e.

$$U_t = UU_{xx} + \lambda_1 UU_x + s_1 U + \tfrac{2}{9}\lambda_1^2 U^2. \qquad (86)$$

This equation has the four-dimensional Lie symmetry [19] and the following family of solutions

$$U = -\frac{9}{2\lambda_1^2}\frac{s_1 + c_1\exp(-\frac{1}{3}\lambda_1 x) + c_2\exp(-\frac{2}{3}\lambda_1 x)}{1 + a_0\exp(-s_1 t)} \qquad (87)$$

that was found in [19] using the Lie method. On the other hand this family of solutions is obtained from (57) for $r = 0, s_0 = 0, q = \frac{2}{9}\lambda_1^2 \neq 0$. Thus, *the non-Lie ansatz* (26) generates *the Lie solutions* (87) for the nonlinear equation (86).

4. Non-Lie exact solutions of the nonlinear equations (9) and (10)

It is easily seen that using substitution (14), any family of the above-found solutions for the nonlinear equation (13) is transformed in a corresponding family of solutions for equations (9) or (10). In table 1, we list the families of the exact solutions of equations (9) and (10) that were found in this paper.

Some of the found solutions can be applied to solving the Dirichlet and the Neumann boundary-value problems for nonlinear heat equations (9) and (10). For example, the solution 1 (see table 1) satisfies the zero boundary condition in the domain $x \in (-\infty, +\infty)$ for any $\alpha < 0, \alpha \neq -1$. In the case of the zero Neumann condition, this solution can be applied for any $\alpha < 0, \alpha \neq -1$ or $\alpha > 1$. Note that the zero Neumann condition (the

Table 1.

	Equation	Family of solutions	Remark
1.	(9) at $\lambda_1 = 0$ $q \neq 0, \alpha \neq -1$	$[\varphi_0(t) + \exp[s_1 t + q\frac{\alpha+2}{\alpha+1}\int \varphi_0(t)\,dt]$ $\times[c_1\exp(-\gamma_1 x) + c_2\exp(\gamma_1 x)]]^{1/\alpha}$	$\varphi_0(t)$ is a solution of (59) $\gamma_1^2 = -\alpha q/(1+\alpha) > 0$
2.	(9) at $\lambda_1 = 0$ $q \neq 0, \alpha \neq -1$	$[\varphi_0(t) + \exp[s_1 t + q\frac{\alpha+2}{\alpha+1}\int \varphi_0(t)\,dt]$ $\times[c_{10}\cos(\lvert\gamma_1\rvert x) + c_{11}\sin(\lvert\gamma_1\rvert x)]]^{1/\alpha}$	$\varphi_0(t)$ is a solution of (63) $\gamma_1^2 = -\alpha q/(1+\alpha) < 0$
3.	(9) at $\lambda_1 = 0$ $q \neq 0, \alpha \neq -1$	$[\varphi_0(t) + c_1\mu(t)^{-\frac{\alpha+2}{\alpha+1}}\exp(\frac{1}{2}s_1 t \pm \gamma_1 x)]^{1/\alpha}$	$\varphi_0(t), \mu(t)$ see in (54), (56) $\gamma_1^2 = -\alpha q/(1+\alpha) > 0$
4.	(9) at $\alpha = -1$ $s_1 \neq 0$, $\lambda_1 = q = 0$	$[\frac{1}{s_1}(-s_0 + c_0 s_1\exp s_1 t - c_1^2\exp 2s_1 t)$ $+c_1 x\exp s_1 t + \varphi_2(t)\exp(\gamma(t)x)]^{-1}$	$\gamma = \frac{s_1}{\gamma_0 s_1 - c_1\exp s_1 t}$, $\varphi_2(t)$ is solution of (71)
5.	(9) at $\alpha = -1$ $s_1 = \lambda_1 = q = 0$	$[c_0 + (s_0 - c_1^2)t + c_1 x +$ $+\varphi_2(t)\exp(\gamma(t)x)]^{-1}$	$\gamma(t) = \frac{1}{\gamma_0 - c_1 t}$ $\varphi_2(t)$ is solution of (72)
6.	(9) at $\alpha = -2$ $s_1 = \lambda_1 = q = 0$	$[c_0\exp(2c_2 t) + c_2(x + \frac{c_1}{2c_2})^2 - \frac{s_0}{2c_2}]^{-1/2}$.	$c_2 \neq 0$
7.	(9) at $\alpha = -2$ $s_1 \neq 0$, $\lambda_1 = q = 0$	$\exp(-\frac{1}{2}s_1 t)[s_0 M(t)\int \frac{dt}{M(t)\exp(s_1 t)}$ $+c_0 M(t) + c_2(x + \frac{c_1}{2c_2})^2]^{-1/2}$	$M(t) = \exp[\frac{2c_2}{s_1}\exp(s_1 t)]$ $c_2 \neq 0$
8.	(9) for $\alpha = -1$ $s_1 = \lambda_1 = q = 0$	$[s_0(t - c_2)\log\lvert t - c_2\rvert$ $+c_0(t - c_2) + \frac{(c_1+x)^2}{2(t-c_2)}]^{-1}$	.
9.	(9) for $\alpha \neq -1, -2$ $s_1 = \lambda_1 = q = 0$	$(c_2 - t)^{-1/\alpha}[-s_0\frac{\alpha+2}{2\alpha+2}(c_2 - t)^2 +$ $c_0\lvert c_2 - t\rvert^{\frac{2}{\alpha+2}} + \frac{\alpha}{2(\alpha+2)}(x + c_1\frac{\alpha+2}{\alpha})^2]^{1/\alpha}$	.
10.	(10) for $\lambda_1^2 - 4q \neq 0$	$\log[\varphi_0(t) + \frac{1}{\mu(t)}[c_1\exp(\frac{1}{2}s_1 t + \gamma_1 x)$ $+c_2\exp(\frac{1}{2}s_1 t + \gamma_2 x)]]^{\frac{1}{\beta}}$	$\varphi_0(t), \mu(t)$ see in (54), (56) $\gamma_{1,2} = \frac{1}{2}(\pm(\lambda_1^2 - 4q)^{1/2} - \lambda_1)$
11.	(10) for $\lambda_1 \neq 0$ $\lambda_1^2 - 4q = 0$	$\log[\varphi_0(t) + \frac{1}{\mu(t)}[c_1\exp(\frac{1}{2}(s_1 t - \lambda_1 x))$ $+c_2 x\exp(\frac{1}{2}(s_1 t - \lambda_1 x))]]^{\frac{1}{\beta}}$	$\varphi_0(t), \mu(t)$ see in (54), (56)
12.	(10) $\lambda_1 \neq 0$, $q = 0$	$\log[\frac{\exp(\frac{1}{2}s_1 t)}{\mu_1(t)}(c_0 + \int \frac{s_0\mu_1(t)}{\exp(\frac{1}{2}s_1 t)}\,dt)$ $+\varphi_1(t)x + \frac{c_2}{\mu_1(t)}\exp(\frac{1}{2}s_1 t - \lambda_1 x)]^{\frac{1}{\beta}}$	$\varphi_1(t), \mu_1(t)$ see in (65), (66)

zero flux on the boundary) is a typical request for describing actual processes in physics, chemistry, and biology. Two examples are considered below.

Example 1. Let us consider the following equation arising in mathematical biology [25]:

$$T_t = [(1 + \lambda_0 T)T_x]_x + \lambda_2 T - \lambda_3 T^2 \qquad (88)$$

that in the case $\lambda_0 = 0, \lambda_2 = \lambda_3$ coincides with the well known Fisher equation [26]

$$T_t = T_{xx} + \lambda_2 T - \lambda_2 T^2. \qquad (89)$$

The known soliton-like solution of the Fisher equation was obtained in [27]. Note that this solution can be also found using the suggested method.

It turns out that the case $\lambda_0 \neq 0$ is very specific. Indeed, equation (88) is reduced to form (9) by the local substitution $1 + \lambda_0 T \to T$. So the solution 1 (see table 1) at $\alpha = 1, c_1 c_2 = 0$ gives the following solutions of equation (88):

$$T = \frac{\lambda_2}{2\lambda_3}\left[1 + \tanh\frac{\lambda_2(t - c_0)}{2}\right] + c_2 \frac{\exp\frac{(2\lambda_3+\lambda_0\lambda_2)t}{4\lambda_0}}{(\cosh\frac{\lambda_2(t-c_0)}{2})^{3/2}} \exp\left(\sqrt{\frac{\lambda_3}{2\lambda_0}}x\right) \tag{90}$$

and

$$T = \frac{\lambda_2}{2\lambda_3}\left[1 + \tanh\frac{\lambda_2(t - c_0)}{2}\right] + c_1 \frac{\exp\frac{(2\lambda_3+\lambda_0\lambda_2)t}{4\lambda_0}}{(\cosh\frac{\lambda_2(t-c_0)}{2})^{3/2}} \exp\left(-\sqrt{\frac{\lambda_3}{2\lambda_0}}x\right) \tag{91}$$

where c_0, c_1, c_2 are arbitrary constants. The solutions of the form (91) have attractive properties: any solution T^* holds the conditions $T^* \to \frac{\lambda_2}{\lambda_3}$ if $t \to \infty$ and $\lambda_3 < \lambda_0\lambda_2$; $T^* \to \frac{\lambda_2}{2\lambda_3}[1 + \tanh\frac{\lambda_2(t-c_0)}{2}] < \frac{\lambda_2}{\lambda_3}$ if $x \to +\infty, \lambda_0\lambda_3 > 0$. Taking into account these properties, we obtain the following theorem.

Theorem 2. The bounded exact solution of the boundary-value problem for the generalized Fisher equation

$$T_t = [(1 + \lambda_0 T)T_x]_x + \lambda_2 T - \lambda_2 T^2 \qquad \lambda_0 > 1, \lambda_2 > 0 \tag{92}$$

with the initial condition

$$T(0, x) = C_0 + C_1 \exp\left(-\sqrt{\frac{\lambda_2}{2\lambda_0}}|x|\right) \tag{93}$$

and the Neumann conditions

$$T_x(t, -\infty) = 0 \qquad T_x(t, +\infty) = 0 \tag{94}$$

is given in the domain $(t, x) \in [0, +\infty) \times (-\infty, +\infty)$ by the formula

$$T = \tfrac{1}{2}\left[1 + \tanh\frac{\lambda_2(t - c_0)}{2}\right] + c_1 \frac{\exp\frac{\lambda_2(2+\lambda_0)t}{4\lambda_0}}{(\cosh\frac{\lambda_2(t-c_0)}{2})^{3/2}} \exp\left(-\sqrt{\frac{\lambda_2}{2\lambda_0}}|x|\right) \tag{95}$$

where $C_0 = \frac{1}{2}[1 + \tanh\frac{-\lambda_2 c_0}{2}]$, $C_1 = c_1(\cosh\frac{-\lambda_2 c_0}{2})^{-3/2}$, and $c_1 > 0$.

Remark 6. Solution (95) is not analytic at $x = 0$. However, the second derivative is well defined at this point since $T_{xx_{|x\to 0+}} = T_{xx_{|x\to 0-}}$. Thus this solution satisfies the equation (92) at $x = 0$ as well.

Example 2. Let us consider the following reaction–diffusion equation with exponential nonlinearities:

$$T_t = (\exp(T)\, T_x)_x + s_0 \exp(-T) + s_1 - s_2 \exp(T). \tag{96}$$

This equation can be applied for describing processes with strong nonlinear diffusion (heat conduction) and reaction (dissipation). In a particular case, using the known series expansion

$$\begin{aligned} \exp(T) &= 1 + T + \frac{T^2}{2} + \cdots \\ \exp(-T) &= 1 - T + \frac{T^2}{2} - \cdots \end{aligned} \tag{97}$$

one can obtain the generalized Fisher equation (92) as some approximation from equation (96).

If one employs the family of solutions 10 (see table 1) at $\beta = 1, \lambda_1 = 0, q = -s_2$, $c_1 = c_2, c_0 = 0$ in the case of equation (96), the following theorem can be formulated.

Theorem 3. The exact solution of the boundary-value problem for the nonlinear equation (96) with the initial condition

$$T(0,x) = \log\left[\frac{s_1}{2s_2} + \frac{c_1}{2}\cosh(\sqrt{s_2}x)\right] \tag{98}$$

and the Neumann conditions

$$T_x(t,0) = 0 \qquad T_x(t,+\infty) = \sqrt{s_2} \tag{99}$$

is given in the domain $(t,x) \in [0,+\infty) \times [0,+\infty)$ by the formula

$$T = \log\left[\frac{s_1}{2s_2} + \frac{\sqrt{D}}{2s_2}\tanh\frac{\sqrt{D}t}{2} + \frac{c_1}{1+\exp(\sqrt{D}t)}\cosh\left(\frac{s_1+\sqrt{D}}{2}t + \sqrt{s_2}x\right)\right] \tag{100}$$

where $c_1 > 0,\ s_1 > 0,\ s_2 > 0,\ D = s_1^2 + 4s_0s_2 > 0$.

Note that in the case $s_2 < 0$ periodic solutions of (96) are obtained and such solutions are potentially interesting for application as well.

5. Discussion

Thus, a constructive method for obtaining new non-Lie ansätze and exact solutions of some classes of nonlinear diffusion equations is developed in this paper. The method is based on the consideration of a fixed nonlinear PDE together with an *additional generating condition* in the form of a linear high-order ODE. With the help of this method new non-Lie ansätze and solutions were obtained for the nonlinear equations (13), (9) and (10). Some of the found solutions can be applied for solving the boundary-value problems for the nonlinear reaction–diffusion equations (9) and (10) and the corresponding examples have been given in section 4.

If additional generating condition does not contain the variable t then it generates the following ansatz with separated variables

$$U = \varphi_0(t)g_0(x) + \cdots + \varphi_{m-1}(t)g_{m-1}(x) \tag{101}$$

that can be generalized (see substitution (2)) to the form

$$A_0(U) = \varphi_0(t)g_0(x) + \cdots + \varphi_{m-1}(t)g_{m-1}(x). \tag{102}$$

In particular cases ansätze (101) and (102) were used for construction of new solutions to nonlinear diffusion equations in the recently published papers [29–33]. The families of solutions of the form

$$A_0(U) = \varphi_0(t) + \varphi_1(t)x + \cdots + \varphi_{m-1}(t)x^{m-1} \qquad m = 3, 4 \text{ or } m = 5 \tag{103}$$

of equation (1) for $B(U) = 0$ with the power and exponential nonlinearities were constructed in these papers (see also the earlier paper [34]). Note that all those solutions can be found using theorem 1 for the case of the additional generating condition

$$\frac{\mathrm{d}^m U}{\mathrm{d}x^m} = 0 \qquad m = 3, 4 \text{ or } m = 5. \tag{104}$$

A generalization of ansatz (103) for the multidimensional case was suggested in [29, 32].

The simplest cases of ansätze (101) and (102) for $m = 2, g_i \neq x^i$ were used for construction of new solutions to nonlinear diffusion equations in [35]. In the cases $m >$

2, $g_i \neq x^i$ the problem of finding solution is not easy. Ansätze (26) for $\gamma_{1,2}(t) =$ constant and (30) for $D(t) =$ constant, $\alpha_2(t) =$ constant were used for finding exact solutions of nonlinear heat equations in the papers [31–33], where the method of linear invariant subspaces was suggested. That method is reduced to finding solutions in the form (101) or (102).

It is clear that any additional generating condition (18) for $m \geqslant 2$ generates the *high-order operator*

$$\alpha_1(t,x)\frac{\partial}{\partial x} + \cdots + \alpha_m(t,x)\frac{\partial^m}{\partial x^m} = 0. \tag{105}$$

The structure of this operator differs from that of any operator of the non-classical symmetry (the conditional symmetry) [36–38] (in [7, 39] one can find a wide list of the references). Indeed, any operator of the non-classical symmetry is *the first-order operator*.

In the recently published papers [40–42] a generalization of the non-classical symmetry is suggested via introduction notions of so-called heir-equations and of the conditional Lie–Bäcklund symmetry. It is easy to check that the majority of the solutions found in [40–42] for nonlinear diffusion equations can be constructed using linear conditions of the form (25) for $\alpha_1, \alpha_2 =$ constant and the corresponding local substitutions for the unknown function $U(t,x)$. On the other hand it means that this method can be connected with the conditional symmetry operators of the high-order.

The suggested method is based on the idea that was applied for the construction of the fundamental solution to the classical multidimensional heat equation in [8]. This solution has been found with the help of an additional linear system of ODEs containing the time t as a parameter. The method enables us to construct solutions of the form

$$A_0(U) = \varphi_0(t)g_0(t,x) + \cdots + \varphi_{m-1}(t)g_{m-1}(t,x) \tag{106}$$

i.e. in a more general form than (101) and (102). For an illustrative example, consider the nonlinear reaction–diffusion equation with a convection term

$$T_t = [T^\alpha T_x]_x + \lambda_1(t)T^\alpha T_x - s_1 T - s_0 T^{1-\alpha} \qquad \alpha \neq 0 \tag{107}$$

that can be interpreted as a generalization of the Fisher and Murray equations [14]. Since it is the particular case of (9) one can reduce this equation to the form

$$U_t = UU_{xx} + \frac{1}{\alpha}U_x^2 + \lambda_1(t)UU_x - \alpha s_1 U - \alpha s_0 \tag{108}$$

using the substitution $U = T^\alpha$. It turns out that equation (108) for $\lambda_1(t) = -(1+\frac{1}{\alpha})\gamma(t)$ is reduced by the ansatz (28) to the following system of ODEs

$$\begin{aligned}
\frac{d\gamma}{dt} &= -\frac{1}{\alpha}\gamma^2\varphi_1 \\
\frac{d\varphi_0}{dt} &= -\left(1+\frac{1}{\alpha}\right)\gamma\varphi_0\varphi_1 - \alpha s_1\varphi_0 + \frac{1}{\alpha}\varphi_1^2 - \alpha s_0 \\
\frac{d\varphi_1}{dt} &= -\alpha s_1\varphi_1 - \left(1+\frac{1}{\alpha}\right)\gamma\varphi_1^2 \\
\frac{d\varphi_2}{dt} &= \left[-\frac{1}{\alpha}\gamma^2\varphi_0 + \left(\frac{1}{\alpha}-1\right)\gamma\varphi_1 - \alpha s_1\right]\varphi_2
\end{aligned} \tag{109}$$

for finding unknown functions $\gamma(t)$ and $\varphi_i, i = 0, 1, 2$. It is easily seen that in this case the function $\gamma(t) \neq$ constant if $\varphi_1 \neq 0$. Solving the system of ODEs (109), we obtain the family of exact solutions that are not the ones with separated variables (101). Note that in the case $\alpha = -1$, the system of ODEs (109) is integrated in terms of elementary functions (see formulae (69)–(72)) and the families of solutions (73) and (74) are found. In the recently published papers [19, 43], one can find similar examples for the nonlinear diffusion *system of equations* describing the process of precipitant-assisted protein crystal growth.

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